Exercise 8

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - y = xe^{2x}, \quad y(0) = 0, \quad y'(0) = 1$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \to \quad y_c' = re^{rx} \quad \to \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 - 1 = 0$$

Solve for r.

$$r = \{-1, 1\}$$

Two solutions to the ODE are e^{-x} and e^x. By the principle of superposition, then,

$$y_c(x) = C_1 e^{-x} + C_2 e^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - y_p = xe^{2x} \tag{2}$$

Since the inhomogeneous term is a polynomial of degree 1 multiplied by an exponential function, the particular solution is $y_p = (A + Bx)e^{2x}$.

$$y_p = (A+Bx)e^{2x} \quad \to \quad y_p' = (B)e^{2x} + 2(A+Bx)e^{2x} \quad \to \quad y_p'' = 2Be^{2x} + 2(B)e^{2x} + 4(A+Bx)e^{2x}$$

Substitute these formulas into equation (2).

$$[2Be^{2x} + 2(B)e^{2x} + 4(A+Bx)e^{2x}] - (A+Bx)e^{2x} = xe^{2x}$$

$$(2B + 2B + 4A - A)e^{2x} + (4B - B)xe^{2x} = xe^{2x}$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\left.\begin{aligned} 2B + 2B + 4A - A &= 0 \\ 4B - B &= 1 \end{aligned}\right\}$$

Solving it yields

$$A = -\frac{4}{9} \quad \text{and} \quad B = \frac{1}{3},$$

which means the particular solution is

$$y_p = \left(-\frac{4}{9} + \frac{1}{3}x\right) e^{2x}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 e^{-x} + C_2 e^{x} + \left(-\frac{4}{9} + \frac{1}{3}x\right) e^{2x}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants. Differentiate it with respect to x.

$$y'(x) = -C_1 e^{-x} + C_2 e^{x} + \left(\frac{1}{3}\right) e^{2x} + 2\left(-\frac{4}{9} + \frac{1}{3}x\right) e^{2x}$$

Apply the boundary conditions to determine C_1 and C_2.

$$y(0) = C_1 + C_2 - \frac{4}{9} = 0$$

$$y'(0) = -C_1 + C_2 - \frac{5}{9} = 1$$

Solving the system yields $C_1 = -5/9$ and $C_2 = 1$. Therefore,

$$y(x) = -\frac{5}{9}e^{-x} + e^{x} + \left(-\frac{4}{9} + \frac{1}{3}x\right) e^{2x}.$$